LECTURE: 2-8 THE DERIVATIVE AS A FUNCTION

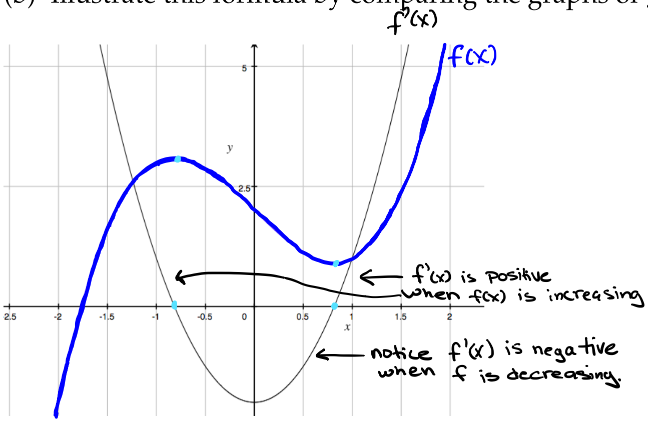
The function
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
is called the **derivative of** f. The value of f' at x can be interpreted geometrically as the slope of the tangent line to f at the point $(x, f(x))$. *Note: f' is called the derivative because it has been derived from f using the limit operation defined above. The domain of f' is the set of all x such that this limit exists and may be smaller than the domain of f.*

Example 1: Let $f(x) = x^3 - 2x + 2$.

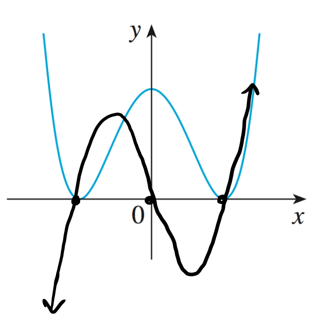
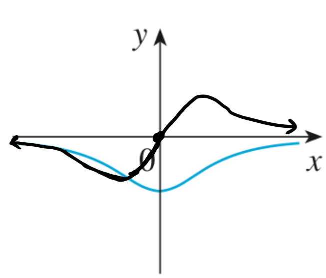
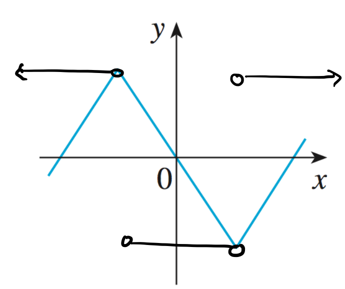
(a) Find a formula for $f'(x)$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^3 - 2(x+h) + 2 - (x^3 - 2x + 2)}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 \cancel{-2x} - 2h \cancel{+2} \cancel{-x^3} \cancel{+2x} \cancel{-2}}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 2h}{h} \\
&= \lim_{h\to 0} (3x^2 + 3xh + h^2 - 2) \\
&= \boxed{3x^2 - 2}
\end{aligned}$$

(b) Illustrate this formula by comparing the graphs of $f(x)$ and $f'(x)$, which are shown below.



Example 2: The graph of f is given below. Use it to sketch the graph of the derivative f'.

Example 3: If $f(x) = \sqrt{x-5}$ find the derivative of f. State the domain of f and f'.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{x+h-5} - \sqrt{x-5}\right)}{(h)} \frac{\left(\sqrt{x+h-5}+\sqrt{x-5}\right)}{\left(\sqrt{x+h-5}+\sqrt{x-5}\right)}$$

$$= \lim_{h\to 0} \frac{(x+h-5)-(x-5)}{h\left(\sqrt{x+h-5}+\sqrt{x-5}\right)}$$

$$= \lim_{h\to 0} \frac{x+h-5-x+5}{h\left(\sqrt{x+h-5}+\sqrt{x-5}\right)}$$

$$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h-5}+\sqrt{x-5}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h-5}+\sqrt{x-5}}$$

$$= \frac{1}{\sqrt{x-5}+\sqrt{x-5}} = \boxed{\frac{1}{2\sqrt{x-5}}}$$

Domain f: $x-5 \geq 0$

$x \geq 5$ $\boxed{[5,\infty)}$

Domain f': $x-5>0$

$x>5$ $\boxed{(5,\infty)}$

Example 4: If $f(x) = \dfrac{2-x}{5+2x}$ find $f'(x)$. State the domain of f and f'.

Domain f: $x \neq -5/2$

Domain f': $x \neq -5/2$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \left(\frac{2-(x+h)}{5+2(x+h)} - \left(\frac{2-x}{5+2x}\right)\right)\frac{1}{h}$$

$$= \lim_{h\to 0} \left[\left(\frac{2-x-h}{5+2x+2h}\right)\frac{(5+2x)}{(5+2x)} - \left(\frac{2-x}{5+2x}\right)\left(\frac{5+2x+2h}{5+2x+2h}\right)\right]\cdot\frac{1}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{10}-\cancel{5x}-5h+\cancel{4x}-\cancel{2x^2}-\cancel{2xh}-(\cancel{10}+\cancel{4x}+4h-\cancel{5x}-\cancel{2x^2}-\cancel{2x}h)}{h(5+2x+2h)(5+2x)}$$

$$= \lim_{h\to 0} \frac{-9h}{h(5+2x+2h)(5+2x)}$$

$$= \lim_{h\to 0} \frac{-9}{(5+2x+2h)(5+2x)}$$

$$= \boxed{\frac{-9}{(5+2x)^2}}$$

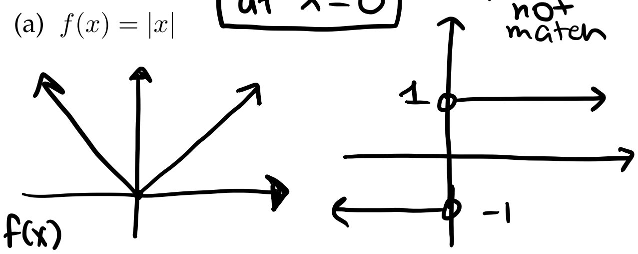
Other Notations for $f'(x)$

given $y = f(x)$, $dy/dx = f'(x)$.

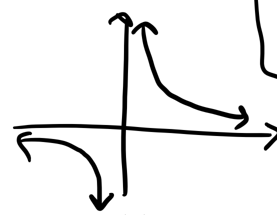
A function f is **differentiable at** a if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or $(a, \infty), (-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Example 5: Where do the following functions fail to be differentiable?

(a) $f(x) = |x|$ $\boxed{\text{at } x=0}$ slopes do not match



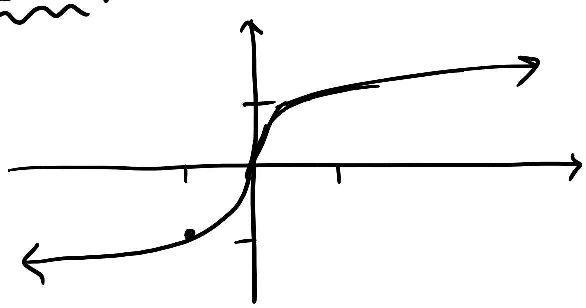
(b) $f(x) = \frac{1}{x}$



at $x=0$ because $f(x)$ is not cts at $x=0$.

Example 6: Where does $f(x) = \sqrt[3]{x}$ fail to be differentiable? Graph $f(x)$ and explain what the behavior of the tangent line is near this point.

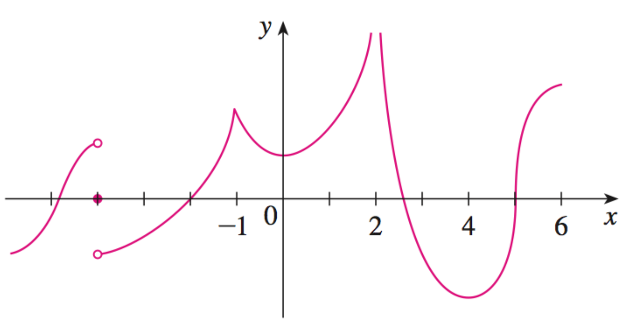
look at:



At $x=0$, the slope of the tangent line is infinite.

Thus $f'(x)$ fails to exist (or f is not differentiable) at $x=0$.

Example 7: A graph of a function $f(x)$ is shown below. State, with reasons, where the function f is not differentiable.



- at $x = -4$, $f(x)$ is not cts
- at $x = -1$, $f(x)$ has corner
- at $x = 2$, $f(x)$ is not cts
- at $x = 5$, $f(x)$ has vertical tangent.

Differentiable Implies Continuous: If f is differentiable at a, then f is continuous at a.

Proof:

We assume f is differentiable, so $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$.

We want to show $\lim_{x\to a} f(x) = f(a)$.

Start w/ $f(x)-f(a) = \frac{f(x)-f(a)}{x-a}(x-a)$.

Now
$$\lim_{x\to a}(f(x)-f(a)) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}(x-a)$$
$$= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}(x-a)$$
$$= f'(a)\cdot 0$$
$$= 0.$$

Thus $\lim_{x\to a}(f(x)-f(a))=0 \Rightarrow \lim_{x\to a} f(x) - \lim_{x\to a} f(a)=0 \Rightarrow \lim_{x\to a} f(x) - f(a) = 0$ and $\lim_{x\to a} f(x)=f(a)$. Thus $f(x)$ is cts. at $x=a$.

Is the converse of this theorem true? That is, if f is continuous at $x = a$ does this imply that f is differentiable at a? Why or why not?

No, $f(x)=|x|$ is cts at $x=0$, but $f'(0)$ DNE.

Higher Derivatives

If f is a differentiable function then its derivative f' is a function, so f' may also have a derivative of its own, denoted by $(f')' = f''$, called the second derivative. Similarly you can also take the derivative of the second derivative, called the third derivative f'''.

Example 8: Given $f(x) = x^3 - 2x + 2$, find and interpret $f''(x)$, $f'''(x)$ and $f^{(4)}(x)$. *(Note: We found $f'(x) = 3x^2 - 2$ in an earlier example.)*

We have $f'(x) = 3x^2-2$

Now
$$f''(x) = \lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h}$$
$$= \lim_{h\to 0}\frac{3(x+h)^2-2-(3x^2-2)}{h}$$
$$= \lim_{h\to 0}\frac{3(x^2+2xh+h^2)-2-3x^2+2}{h}$$
$$= \lim_{h\to 0}\frac{6xh+3h^2}{h}$$
$$= \lim_{h\to 0}(6x+3h)$$
$$= \boxed{6x}$$

$$f'''(x) = \lim_{h\to 0}\frac{f''(x+h)-f''(x)}{h}$$
$$= \lim_{h\to 0}\frac{6(x+h)-6x}{h}$$
$$= \lim_{h\to 0}\frac{6x+6h-6x}{h}$$
$$= \lim_{h\to 0}\frac{6h}{h}$$
$$= \boxed{6}$$

$$\boxed{f^{(4)}(x)=0}$$